XLII. *A Method of extending* Cardan's *Rule for resolving one Case of a Cubick Equation of this Form,* $x^3 * - qx = r$, *to the other Case of the same Equation, which it is not naturally fitted to solve, and which is therefore often called* the irreducible Case. *By* Francis Maseres, *Esq. F. R. S. Cursitor Baron of the Exchequer.*

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ARTICLE I.

IT is well known to all persons conversant with algebra, that CARDAN's rule for resolving the cubick equation $x^3 - qx = r$ is only fitted to resolve it when $\frac{rr}{4}$ is equal to, or greater than, $\frac{q^3}{27}$, or when r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and that it is of no use in the resolution of the other case of this equation, in which r is of any magnitude less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$. For in this case $\frac{rr}{4} - \frac{q^3}{27}$ becomes (according to the usual language of algebraists) a negative quantity, and consequently its square-root becomes

comes impoſſible, and the expreſſion given by CARDAN's rule for the value of x (which is either $\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}+\frac{q}{3\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}}$ or $\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}+\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}$), involves in it the impoſſible quantity $\sqrt{\frac{rr}{4}-\frac{q^3}{27}}$, and therefore is unintelligible and uſeleſs: or, according to what appears to me a more correct way of ſpeaking (who never could form any idea of a negative quantity, and never underſtand by the ſign $-$ any thing more than the ſubtraction of a leſſer quantity from a greater), the quantity $\frac{rr}{4}-\frac{q^3}{27}$ becomes itſelf impoſſible, or the ſuppoſition that $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, (which is one of the foundations of CARDAN's rule), is no longer true, and conſequently the rule itſelf, which is built upon it, can no longer take place.

2. Nevertheleſs it is poſſible, by the help of Sir ISAAC NEWTON's binomial theorem, to extend this rule to this latter caſe, in which $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, and which it is not of itſelf fitted to reſolve; or, to ſpeak with more accuracy, it is poſſible to derive from the expreſſion of the value of x given by CARDAN's rule for the reſolution of the equation $x^3-qx=r$ in the firſt caſe, in which $\frac{rr}{4}$ is greater

greater than $\frac{q^3}{27}$, another expreſſion ſomewhat different from the former, that ſhall exhibit the true value of x in the ſecond caſe, in which $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, provided it be not leſs than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$: and this without any mention of either impoſſible, or negative, quantities. To ſhew how this may be effected, is the deſign of the following pages.

3. That the whole of this matter may be ſeen at one view, it will be convenient to ſet forth the foundation and inveſtigation of CARDAN's rule for reſolving the equation $x^3 - qx = r$, when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$; which may be done as follows.

The Inveſtigation of Cardan's *Rule for reſolving the Cubick Equation* $x^3 * - qx = r$, *when* $\frac{rr}{4}$ *is greater than* $\frac{q^3}{27}$.

4. Previouſly to the inveſtigation of this rule, it will be proper to make the following obſervations.

OBS. 1. In the equation $x^3 - qx = r$ (which is a propoſition affirming that x^3 is greater than qx, and that the exceſs is equal to r) xx muſt always be greater than q, and x than $\sqrt{q}$.

OBS.

OBS. 2. While x increases from $\sqrt{q}$ *ad infinitum*, x^3 will increase continually from $q\sqrt{q}$ *ad infinitum*, and qx will increase continually from the same quantity $q\sqrt{q}$ *ad infinitum*.

OBS. 3. Also, while x increases from $\sqrt{q}$ *ad infinitum*, the excess of x^3 above qx will increase continually from nothing *ad infinitum*, without ever decreasing. For, if we put $\dot{x}$ to denote the increment which x receives in any given time, either small or great, $q\dot{x}$ will be the increment which qx will receive in the same time, and $3x^2\dot{x}+3x\dot{x}^2+\dot{x}^3$ will be the increment of x^3 in the same time. Now, since xx is always greater than q during the whole increase of x from being equal to $\sqrt{q}$ *ad infinitum*, $xx\times\dot{x}$ will be greater than $q\dot{x}$ during that whole increase. Therefore, *à fortiori*, $3x^2\dot{x}+3x\dot{x}^2+\dot{x}^3$ (which is more than triple of $xx\times\dot{x}$) will be greater than $q\dot{x}$; that is, the increment of x^3 will be greater than the contemporary increment of qx during all the increase of x. Consequently the excess of x^3 above qx, or the compound quantity x^3-qx, will continually increase, without ever decreasing, while x increases from $\sqrt{q}$ to any greater magnitude.

OBS. 4. Since the compound quantity x^3-qx increases continually at the same time as x increases; and, when

 x is

x is equal to $\frac{2\sqrt{q}}{\sqrt{3}}$, x^3-qx is $\left(=\frac{8q\sqrt{q}}{3\sqrt{3}}-\frac{2q\sqrt{q}}{\sqrt{3}}=\frac{8q\sqrt{q}}{3\sqrt{3}}-\frac{6q\sqrt{q}}{3\sqrt{3}}\right)=\frac{2q\sqrt{q}}{3\sqrt{3}}$, it follows that, if x is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, the compound quantity x^3-qx will be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and, if x is less than $\frac{2\sqrt{q}}{\sqrt{3}}$, the said compound quantity will be less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and, *è converso*, if the compound quantity x^3-qx, or, its equal, the absolute term r, is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the value of x will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$; and, if x^3-qx, or r, is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the value of x will be less than $\frac{2\sqrt{q}}{\sqrt{3}}$; or, if $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, x will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, and, if $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, x will be less than $\frac{2\sqrt{q}}{\sqrt{3}}$.

Obs. 5. When r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and consequently (by the last observation) x is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, xx will be greater than $\frac{4q}{3}$, and $\frac{xx}{4}$ will be greater than $\frac{q}{3}$. But $\frac{xx}{4}$ is the square of $\frac{x}{2}$. Therefore when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{3}{27}$, the square of half x will be greater than $\frac{q}{3}$. But (by Euclid's Elements, Book II. Prop. V.) it is always possible to divide a line, as x, into two unequal parts in such a proa proportion that the rectangle under its parts ſhall be equal to any quantity that is leſs than the ſquare of its half. Therefore, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it is poſſible to divide the line, or root, x into two unequal parts of ſuch magnitudes that their rectangle, or product, ſhall be equal to $\frac{q}{3}$. This obſervation is the foundation of CARDAN's rule for the reſolution of the equation $x^3-qx=r$ in the firſt caſe of that equation, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$; the inveſtigation of which is as follows.

PROBLEM.

5. *To reſolve the Equation $x^3 * -qx=r$, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.*

SOLUTION.

Since r is ſuppoſed to be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and conſequently (by Obſ. 5.) $\frac{xx}{4}$ is greater than $\frac{q}{3}$, it is poſſible for x to be divided into two unequal parts of ſuch magnitudes that their rectangle, or product, ſhall be equal to $\frac{q}{3}$. Let it be conceived to be ſo divided; and let the greater of the two parts be called a, and the leſſer b. Then will ab be $=\frac{q}{3}$, and conſequently $3ab$ will be $=q$, and $3ab\times\overline{a+b}$ will be $=q\times\overline{a+b}$.

Now, ſince $a+b$ is equal to x, we ſhall have $a^3+3aab+3abb+b^3=x^3$, and $q\times\overline{a+b}=qx$. Therefore x^3-qx will be $=a^3+3aab+3abb+b^3-q\times\overline{a+b}=a^3+3ab\times\overline{a+b}+b^3-q\times\overline{a+b}$; that is (becauſe $3ab\times\overline{a+b}$ is $=q\times\overline{a+b}$) x^3-qx will be $=a^3+b^3$. Therefore r (which is $=x^3-qx$) will be $=a^3+b^3$.

But, ſince $3ab$ is $=q$, we ſhall have $b=\frac{q}{3a}$, and $b^3=\frac{q^3}{27a^3}$. Therefore a^3+b^3 is $=a^3+\frac{q^3}{27a^3}$, and r (which is $=a^3+b^3$) is $=a^3+\frac{q^3}{27a^3}$. Therefore ra^3 is $=a^6+\frac{q^3}{27}$, and ra^3-a^6 is $=\frac{q^3}{27}$.

But ra^3-a^6 is the product of the multiplication of $r-a^3$ into a^3, which are together equal to r. Therefore (by El. II. 5.) ra^3-a^6 muſt be leſs than the ſquare of half r, that is, than $\frac{rr}{4}$, and conſequently may be ſubtracted from it. Let it, and its equal $\frac{q^3}{27}$, be ſo ſubtracted. And we ſhall have $\frac{rr}{4}-ra^3+a^6=\frac{rr}{4}-\frac{q^3}{27}$. Therefore the ſquare-root of $\frac{rr}{4}-ra^3+a^6$ will be equal to $\sqrt{\frac{rr}{4}-\frac{q^3}{27}}$. But the ſquare-root of $\frac{rr}{4}-ra^3+a^6$ is the difference of $\frac{r}{2}$ and a^3, that

that is, either $\frac{r}{2} - a^3$ or $a^3 - \frac{r}{2}$, according as $\frac{r}{2}$ or a^3 is the greater quantity. But it has appeared above that a^3 and b^3 together are equal to r; and a is ſuppoſed to be greater than b, and conſequently a^3 is greater than b^3. Therefore a^3 muſt be greater, and b^3 leſs, than $\frac{r}{2}$. Therefore $a^3 - \frac{r}{2}$ is the difference of a^3 and $\frac{r}{2}$, and conſequently is the ſquare-root of the quantity $\frac{rr}{4} - ra^3 + a^6$. Therefore $a^3 - \frac{r}{2}$ is $= \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and a^3 is $= \frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Conſequently a is $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$. But b has been ſhewn to be $= \frac{q}{3a}$. Therefore b is $= \frac{q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$; and conſequently $a + b$, or x, is $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$.

Q. E. I.

6. This expreſſion may be rendered more ſimple by ſubſtituting the ſingle letter s in it inſtead of $\sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. For then it will be $\sqrt[3]{\frac{r}{2} + s} + \frac{q}{3\sqrt[3]{\frac{r}{2} + s}}$.

Synthetick Demonſtration of the Truth of the foregoing Solution.

7. That this expreſſion is equal to x in the equation

$x^3-qx=r$ will appear by ſubſtituting it inſtead of x in the compound quantity x^3-qx, which will thereby be ſeen to be equal to r, as it ought to be.

This may be done in the manner following.

Since x is $=\sqrt[3]{\frac{r}{2}+s}+\frac{q}{3\sqrt[3]{\frac{r}{2}+s}}$, or $\left|\frac{r}{2}+s\right|^{\frac{1}{3}}+\frac{q}{3\times\left|\frac{r}{2}+s\right|^{\frac{1}{3}}}$, we ſhall have $x^3=\frac{r}{2}+s+3\times\left|\frac{r}{2}+s\right|^{\frac{2}{3}}\times\frac{q}{3\times\left|\frac{r}{2}+s\right|^{\frac{1}{3}}}+3\times\left|\frac{r}{2}+s\right|^{\frac{1}{3}}\times\frac{qq}{9\times\left|\frac{r}{2}+s\right|^{\frac{2}{3}}}+\frac{q^3}{27\times\frac{r}{2}+s}=\frac{r}{2}+s+q\times\left|\frac{r}{2}+s\right|^{\frac{1}{3}}+\frac{qq}{3\times\left|\frac{r}{2}+s\right|^{\frac{1}{3}}}+\frac{q^3}{\frac{27r}{2}+27s}$; and $qx=q\times\left|\frac{r}{2}+s\right|^{\frac{1}{3}}+\frac{qq}{3\times\left|\frac{r}{2}+s\right|^{\frac{1}{3}}}$; and conſequently $x^3-qx=\frac{r}{2}+s+\frac{q^3}{\frac{27r}{2}+27s}=\frac{r}{2}+s+\frac{q^3}{\frac{27r}{2}+\frac{54s}{2}}=\frac{r}{2}+s+\frac{q^3}{\frac{27r+54s}{2}}=\frac{r}{2}+s+\frac{2q^3}{27r+54s}$.

Now ss, or $\frac{rr}{4}-\frac{q^3}{27}$, is $\frac{27rr-4q^3}{108}=\frac{27rr-4q^3}{36\times 3}$; or, if we put $mm=27rr-4q^3$, we ſhall have ss, or $\frac{rr}{4}-\frac{q^3}{27}$, $=\frac{mm}{36\times 3}$, and $s=\frac{m}{6\sqrt{3}}$. Therefore $\frac{2q^3}{27r+54s}$ is $=\frac{2q^3}{27r+\frac{54\times m}{6\sqrt{3}}}=\frac{2q^3}{\frac{6\times 27\times\sqrt{3}\times r+54m}{6\sqrt{3}}}=\frac{12\sqrt{3}\times q^3}{6\times 27\sqrt{3}\times r+54m}=\frac{2\sqrt{3}\times q^3}{27\sqrt{3}\times r+9m}$. Therefore $s+\frac{2q^3}{27r+54s}$ is $=\frac{m}{6\sqrt{3}}+\frac{2\sqrt{3}\times q^3}{27\sqrt{3}\times r+9m}=\frac{27\sqrt{3}\times rm+9mm+36q^3}{6\times 27\times 3r+6\times 9\times\sqrt{3}\times m}=\frac{3\sqrt{3}\times rm+mm+4q^3}{54r+6\sqrt{3}\times m}=\frac{3\sqrt{3}\times rm+27rr-4q^3+4q^3}{54r+6\sqrt{3}\times m}=\frac{3\sqrt{3}\times rm+27rr}{54r+6\sqrt{3}\times m}=\frac{\sqrt{3}\times rm+9rr}{18r+2\sqrt{3}\times m}$; and $\frac{r}{2}+$

$s+$

$s+\frac{2q^3}{27r+54s}$ is $=\frac{r}{2}+\frac{\sqrt{3}\times rm+9rr}{18r+2\sqrt{3}\times m}=\frac{18rr+2\sqrt{3}\times rm+2\sqrt{3}\times rm+18rr}{36r+4\sqrt{3}\times m}=\frac{36rr+4\sqrt{3}\times rm}{36r+4\sqrt{3}\times m}=r$. But we have before shewn that x^3-qx is $=\frac{r}{2}+s+\frac{2q^3}{27r+54s}$. Therefore x^3-qx is $=r$, and consequently $\sqrt[3]{\frac{r}{2}+s}+\frac{q}{3\sqrt[3]{\frac{r}{2}+s}}$ is the true value of x in the cubick equation $x^3-qx=r$. Q. E. D.

Two other Expressions for the Root of the foregoing Equation.

8. Two other expressions may be found for the root of this equation by resuming the investigation contained in Art. 5. The first of these expressions is $\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}+\frac{q}{3\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}}$, or (if we put ss, as before, $=\frac{rr}{4}-\frac{q^3}{27}$,) $\sqrt[3]{\frac{r}{2}-s}+\frac{q}{3\sqrt[3]{\frac{r}{2}-s}}$. The other expression is $\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}+\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}$, or $\sqrt[3]{\frac{r}{2}+s}+\sqrt[3]{\frac{r}{2}-s}$. These expressions are to be found in the following manner.

Investigation of the said Expressions.

9. In Art. 5. we supposed the line x to be divided into

to two unequal parts a and b, of which a was ſuppoſed to be the greater; and we firſt found the value of the greater part a, and then determined that of the leſſer part b from its relation to a, which is expreſſed by the equation $3ab = q$. But we may with the ſame eaſe firſt determine the value of the leſſer part b, and then derive from it that of the greater part a; which would produce the firſt of the two expreſſions of the value of x mentioned in the laſt article. This may be done as follows.

Since it has been ſhewn in Art. 5. that r is $= a^3 + b^3$, and $3ab$ is $= q$, and conſequently a is $= \frac{q}{3b}$, and a^3 to $\frac{q^3}{27b^3}$, it follows that r will be $= \frac{q^3}{27b^3} + b^3$. Therefore rb^3 is $= \frac{q^3}{27} + b^6$, and (ſubtracting b^6 from both ſides) $rb^3 - b^6$ is $= \frac{q^3}{27}$. Therefore (ſubtracting both ſides from $\frac{rr}{4}$, than which they are evidently leſs), we ſhall have $\frac{rr}{4} - rb^3 + b^6 = \frac{rr}{4} - \frac{q^3}{27}$. Therefore the ſquare-root of $\frac{rr}{4} - rb^3 + b^6$ will be $= \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. But the ſquare-root of $\frac{rr}{4} - rb^3 + b^6$ is the difference of the quantities $\frac{r}{2}$ and b^3, that is (becauſe b^3 is the leſſer part of $a^3 + b^3$, or r, and conſequently is leſs than the half of it, or $\frac{r}{2}$), it is $= \frac{r}{2} - b^3$. Therefore $\frac{r}{2} - b^3$ is $= \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Therefore (adding b^3 to both ſides) $\frac{r}{2}$ will be $= b^3 +$

$\checkmark$

$\sqrt{\frac{rr}{4}-\frac{q^3}{27}}$, and (subtracting $\sqrt{\frac{rr}{4}-\frac{q^3}{27}}$ from both sides) b^3 will be $=\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}$. Therefore b is $=\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}$, and a $(=\frac{q}{3b})$ is $=\frac{q}{3\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}}$, and consequently $b+a$, or $a+b$, or x, is $=\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}+\frac{q}{3\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}}$, or (if we put $ss=\frac{rr}{4}-\frac{q^3}{27}$) $\sqrt[3]{\frac{r}{2}-s}+\frac{q}{3\sqrt[3]{\frac{r}{2}-s}}$. Q. E. I.

Synthetick demonstration of the truth of the foregoing expression.

10. Here again we may demonstrate synthetically, that this expression is equal to the true value of x in the proposed equation $x^3-qx=r$, by substituting it for x in the left-hand side of that equation. For, if we make that substitution, we shall find that the value of x^3-qx thence arising will be equal to r. This may be done in the manner following.

If x is $=\sqrt[3]{\frac{r}{2}-s}+\frac{q}{3\sqrt[3]{\frac{r}{2}-s}}$, or $\overline{\frac{r}{2}-s}|^{\frac{1}{3}}+\frac{q}{3\times\overline{\frac{r}{2}-s}|^{\frac{1}{3}}}$, we shall have $x^3=\frac{r}{2}-s+3\times\overline{\frac{r}{2}-s}|^{\frac{2}{3}}\times\frac{q}{3\times\overline{\frac{r}{2}-s}|^{\frac{1}{3}}}+3\times\overline{\frac{r}{2}-s}|^{\frac{1}{3}}\times$

$\frac{qq}{9\times\overline{\frac{r}{2}-s}|^{\frac{2}{3}}}+\frac{q^3}{27\times\overline{\frac{r}{2}-s}}=\frac{r}{2}-s+q\times\overline{\frac{r}{2}-s}|^{\frac{1}{3}}+\frac{qq}{3\times\overline{\frac{r}{2}-s}|^{\frac{1}{3}}}+$ $\frac{q^3}{27\times\overline{\frac{r}{2}-s}}$, and $qx=q\times\overline{\frac{r}{2}-s}|^{\frac{1}{3}}+\frac{qq}{3\times\overline{\frac{r}{2}-s}|^{\frac{1}{3}}}$, and consequently $x^3-qx=\frac{r}{2}-s+\frac{q^3}{27\times\overline{\frac{r}{2}-s}}=\frac{r}{2}-s+\frac{q^3}{\frac{27r}{2}-27s}=\frac{r}{2}-s$ $+\frac{q^3}{\frac{27r}{2}-\frac{54s}{2}}=\frac{r}{2}-s+\frac{2q^3}{27r-54s}$. Now ss, or $\frac{rr}{4}-\frac{q^3}{27}$ is $=\frac{27rr-4q^3}{108}=$ $\frac{27rr-4q^3}{36\times 3}$. Therefore if we put $mm=27rr-4q^3$, we shall have $ss=\frac{mm}{36\times 3}$, and $s=\frac{m}{6\sqrt{3}}$. Therefore $\frac{2q^3}{27r-54s}$ is $=$

$$\frac{2q^3}{27r-\frac{54m}{6\sqrt{3}}}=\frac{2q^3}{\frac{6\times 27\times\sqrt{3}\times r-54m}{6\sqrt{3}}}=\frac{12\sqrt{3}\times q^3}{6\times 27\times\sqrt{3}\times r-54m}=\frac{2\sqrt{3}\times q^3}{27\times\sqrt{3}\times r-9m}.$$

Therefore $\frac{r}{2}-s+\frac{2q^3}{27r-54s}$ is $=\frac{r}{2}-\frac{m}{6\sqrt{3}}+\frac{2\sqrt{3}\times q^3}{27\times\sqrt{3}\times r-9m}=\frac{r}{2}$ $\frac{-27\times\sqrt{3}\times rm+9mm+36q^3}{6\times 27\times 3r-54\sqrt{3}\times m}=\frac{r}{2}\,\frac{-3\sqrt{3}\times rm+mm+4q^3}{54r-6\sqrt{3}\times m}=\frac{r}{2}$ $\frac{-3\sqrt{3}\times rm+27rr-4q^3+4q^3}{54r-6\sqrt{3}\times m}=\frac{r}{2}\,\frac{-3\sqrt{3}\times rm+27rr}{54r-6\sqrt{3}\times m}=\frac{54rr-6\sqrt{3}\times rm-6\sqrt{3}\times rm+54rr}{108r-12\sqrt{3}\times m}$ $=\frac{108rr-12\sqrt{3}\times rm}{108r-12\sqrt{3}\times m}=r$. But it has been before shewn that x^3-qx is $=\frac{r}{2}-s+\frac{2q^3}{27r-54s}$. Therefore x^3-qx is $=r$; and $\sqrt[3]{\frac{r}{2}-s}+\frac{q}{3\sqrt[3]{\frac{r}{2}-s}}$ is the true value of x in the cubick equation $x^3-qx=r$. Q. E. D.

Investigation

Investigation of the third expression of the value of the root x.

11. The third expression for the value of x, or the last of the two mentioned in Art. 8. to wit, $\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}+\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}$, or $\sqrt[3]{\frac{r}{2}+s}+\sqrt[3]{\frac{r}{2}-s}$, may be obtained as follows.

Since a^3+b^3 is $=r$, it follows that b^3 will be $=r-a^3$. But a^3 is shewn in Art. 5. to be $=\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}$. Therefore $r-a^3$ is $=r-\overline{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}=\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}$. Consequently b^3 is $=\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}$, and b is $=\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}$, and $a+b$, or x, is $=\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}+\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}$, or (putting ss, as before, $=\frac{rr}{4}-\frac{q^3}{27}$,) $\sqrt[3]{\frac{r}{2}+s}+\sqrt[3]{\frac{r}{2}-s}$. Q. E. I.

Synthetick demonstration of the truth of the said third expression.

12. Here again we may demonstrate synthetically, that this expression is equal to the true value of x in the equation $x^3-qx=r$, by substituting it for x in the left-hand side of the said equation. For, if we make that substitution, we shall find that the value of x^3-qx,

 thence

thence arising, will be equal to r. This may be done in the following manner.

If x is $=\sqrt[3]{\frac{r}{2}+s}+\sqrt[3]{\frac{r}{2}-s}$, or $\left(\frac{r}{2}+s\right)^{\frac{1}{3}}+\left(\frac{r}{2}-s\right)^{\frac{1}{3}}$, we shall have

$$x^3=\frac{r}{2}+s+3\times\left(\frac{r}{2}+s\right)^{\frac{2}{3}}\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}+3\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}\times\left(\frac{r}{2}-s\right)^{\frac{2}{3}}+\frac{r}{2}-s=r+3\times\left(\frac{r}{2}+s\right)^{\frac{2}{3}}\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}+3\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}\times\left(\frac{r}{2}-s\right)^{\frac{2}{3}}=r+3\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}+3\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}=r+3\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}\times\left(\frac{rr}{4}-ss\right)^{\frac{1}{3}}+3\times\left(\frac{rr}{4}-ss\right)^{\frac{1}{3}}\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}=r+3\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}\times\left(\frac{rr}{4}-\frac{rr}{4}+\frac{q^3}{27}\right)^{\frac{1}{3}}+3\times\left(\frac{rr}{4}-\frac{rr}{4}+\frac{q^3}{27}\right)^{\frac{1}{3}}\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}=r+3\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}\times\left(\frac{q^3}{27}\right)^{\frac{1}{3}}+3\times\left(\frac{q^3}{27}\right)^{\frac{1}{3}}\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}=r+3\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}\times\frac{q}{3}+3\times\frac{q}{3}\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}=r+q\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}+q\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}.$$

And qx will be $=q\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}+q\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}$. Therefore x^3-qx will be $=r+q\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}+q\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}-q\times\left(\frac{r}{2}+s\right)^{\frac{1}{3}}-q\times\left(\frac{r}{2}-s\right)^{\frac{1}{3}}=r$; and consequently $\left(\frac{r}{2}+s\right)^{\frac{1}{3}}+\left(\frac{r}{2}-s\right)^{\frac{1}{3}}$, or $\sqrt[3]{\frac{r}{2}+s}+\sqrt[3]{\frac{r}{2}-s}$, is the true value of x in the cubick equation $x^3-qx=r$.

13. N.B. I do not remember to have seen these substitutions, or synthetical demonstrations of the truth of the expressions given by CARDAN's rule, in any book of algebra.

14. *An Example of the Resolution of a Cubick Equation of the aforesaid Form,* $x^3 - qx = r$, *by means of each of the three foregoing Expressions.*

I will here insert a single example of a numeral equation of the foregoing form, $x^3 - qx = r$, resolved by each of the three expressions above-mentioned, in order to shew that they will all three bring out the same number for its root..

Let it therefore be required to find the value of x in the cubick equation $x^3 - 3x = 18$.

15. In this equation q is $= 3$, and r is $= 18$. Therefore $\sqrt{q}$ is $= \sqrt{3}$, and $\frac{2q\sqrt{q}}{3\sqrt{3}}$ is $= \frac{2 \times 3\sqrt{3}}{3\sqrt{3}} = 2$, which is greatly less than 18, or r. Therefore this equation comes under the above-mentioned rule, and may be resolved by either of the foregoing expressions.

Resolution of the equation $x^3 - 3x = 18$ *by the first of the said expressions.*

16. The first of those expressions is $\sqrt[3]{\frac{r}{2} + s} + \frac{q}{3\sqrt[3]{\frac{r}{2} + s}}$, in which s stands for $\sqrt{\frac{rr}{4} - \frac{q^3}{27}}$.

Now

Now, since q is $=3$, $\frac{q}{3}$ will be $=\frac{3}{3}=1$, and consequently $\frac{q^3}{27}$, or the cube of $\frac{q}{3}$, will also be $=1$. And, since r is $=18$, we shall have $\frac{r}{2}=9$, and $\frac{rr}{4}=81$, and consequently $\frac{rr}{4}-\frac{q^3}{27}=81-1=80$; that is, ss will be $=80$. Therefore s is $=\sqrt{80}=\sqrt{16}\times\sqrt{5}=4\sqrt{5}$; and $\frac{r}{2}+s$ is $=9+4\sqrt{5}=\frac{72+32\sqrt{5}}{8}=\frac{27+27\sqrt{5}+45+5\sqrt{5}}{8}$; and consequently $\sqrt[3]{\frac{r}{2}+s}$ is $=\frac{3+\sqrt{5}}{2}$. Therefore $3\times\sqrt[3]{\frac{r}{2}+s}$ is $=\frac{3\times\overline{3+\sqrt{5}}}{2}$, and $\frac{q}{3\sqrt[3]{\frac{r}{2}+s}}$ is $=3\times\frac{2}{3\times\overline{3+\sqrt{5}}}=\frac{2}{3+\sqrt{5}}$; and $\sqrt[3]{\frac{r}{2}+s}+\frac{q}{3\sqrt[3]{\frac{r}{2}+s}}$ is $=\frac{3+\sqrt{5}}{2}+\frac{2}{3+\sqrt{5}}=\frac{\overline{3+\sqrt{5}}\times\overline{3+\sqrt{5}}+4}{2\times\overline{3+\sqrt{5}}}=\frac{9+6\sqrt{5}+5+4}{2\times\overline{3+\sqrt{5}}}=\frac{18+6\sqrt{5}}{2\times\overline{3+\sqrt{5}}}=\frac{6\times\overline{3+\sqrt{5}}}{2\times\overline{3+\sqrt{5}}}=\frac{6}{2}=3$. Therefore 3 is the value of x in the equation $x^3-3x=18$. And so we shall find it to be upon trial: for, if x is taken $=3$, we shall have $x^3=27$, and $3x=3\times3=9$, and $x^3-3x=27-9=18$. And thus we see that the first of the three foregoing expressions, to wit, $\sqrt[3]{\frac{r}{2}+s}+\frac{q}{3\sqrt[3]{\frac{r}{2}+s}}$, has given us the true value of x in this equation.

Resolution

Resolution of the same equation by the second and third of the foregoing expressions.

17. We are now to resolve the same equation $x^3-3x=18$ by means of the two other expressions, to wit, $\sqrt[3]{\frac{r}{2}-s}+\frac{q}{3\sqrt[3]{\frac{r}{2}-s}}$, and $\sqrt[3]{\frac{r}{2}+s}+\sqrt[3]{\frac{r}{2}-s}$.

Now, since r is $=18$, and s has been shewn to be $=\sqrt{80}$, or $4\sqrt{5}$, we shall have $\frac{r}{2}-s=9-4\sqrt{5}=\frac{72-32\sqrt{5}}{8}=\frac{27-27\sqrt{5}+45-5\sqrt{5}}{8}$, and $\sqrt[3]{\frac{r}{2}-s}=\frac{3-\sqrt{5}}{2}$. Therefore $3\sqrt[3]{\frac{r}{2}-s}$ is $=\frac{3\times\overline{3-\sqrt{5}}}{2}$, and $\frac{q}{3\sqrt[3]{\frac{r}{2}-s}}$ is $=\frac{3}{\frac{3\times\overline{3-\sqrt{5}}}{2}}=3\times\frac{2}{3\times\overline{3-\sqrt{5}}}=\frac{2}{3-\sqrt{5}}$. Consequently $\sqrt[3]{\frac{r}{2}-s}+\frac{q}{3\sqrt[3]{\frac{r}{2}-s}}$ is $=\frac{3-\sqrt{5}}{2}+\frac{2}{3-\sqrt{5}}=\frac{\overline{3-\sqrt{5}}\times\overline{3-\sqrt{5}}+4}{2\times\overline{3-\sqrt{5}}}=\frac{9-6\sqrt{5}+5+4}{2\times\overline{3-\sqrt{5}}}=\frac{18-6\sqrt{5}}{2\times\overline{3-\sqrt{5}}}=\frac{9-3\sqrt{5}}{3-\sqrt{5}}=\frac{3\times\overline{3-\sqrt{5}}}{3-\sqrt{5}}=3$. Therefore x is $=3$, as it was found to be by the first expression.

18. The third expression $\sqrt[3]{\frac{r}{2}+s}+\sqrt[3]{\frac{r}{2}-s}$ is in the present case $=\frac{3+\sqrt{5}}{2}+\frac{3-\sqrt{5}}{2}=\frac{6}{2}=3$. Therefore by this expression, as well as by both the former, the value of x in the equation $x^3-3x=18$ comes out to be 3.

19. *Note.* The foregoing method of reſolving the cubick equation $x^3 - qx = r$, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and a like method of reſolving the cubick equation $x^3 + qx = r$ (which holds good in all caſes, whatever be the magnitudes of q and r), are uſually known by the name of CARDAN's *rules*, becauſe they were firſt publiſhed by him in his treatiſe of algebra, intitled, *Ars magna, quam vulgò Coſſam vocant, ſeu regulas Algebraicas*, in the year 1545, although, as he himſelf informs us, they were firſt found out by one SCIPIO FERREUS of Bononia. See WALLIS's algebra, Chap. XIII.

Of the ſecond caſe of the cubick equation $x - qx = r$; in which r is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, and which cannot be reſolved by CARDAN's *rule.*

20. The remaining caſe of the cubick equation $x^3 - qx = r$, in which r is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, and which conſequently cannot be reſolved by the rules above-mentioned, has, upon that account, obtained amongſt algebraiſts the name of the *irreducible caſe:* at leaſt it is often called by the French writers of algebra *le cas irréductible.* The object of the remaining pages of

of this paper is to ſhew how, by the help of Sir ISAAC NEWTON's famous binomial theorem, the foregoing ſolution of the other, or firſt, caſe of this equation may be, as it were, extended to this latter caſe, or, rather, may be made the means of diſcovering, by a very peculiar train of reaſoning, another ſolution, that ſhall be adapted to it.

21. By the binomial theorem it appears that the cube-root of the binomial quantity $a+b$ (in which a is ſuppoſed to be greater than b) is equal to the following infinite ſeries, to wit, $a^{\frac{1}{3}}+\frac{a^{\frac{1}{3}}b}{3a}-\frac{a^{\frac{1}{3}}b^2}{9a^2}+\frac{5a^{\frac{1}{3}}b^3}{81a^3}-\frac{10a^{\frac{1}{3}}b^4}{243a^4}+\frac{22a^{\frac{1}{3}}b^5}{729a^5}-\frac{154a^{\frac{1}{3}}b^6}{6561a^6}+\frac{2618a^{\frac{1}{3}}b^7}{137,781a^7}-$ &c. or to $a^{\frac{1}{3}}\times$ the infinite ſeries $1+\frac{b}{3a}-\frac{b^2}{9a^2}+\frac{5b^3}{81a^3}-\frac{10b^4}{243a^4}+\frac{22b^5}{729a^5}-\frac{154b^6}{6561a^6}+\frac{2618b^7}{137,781a^7}-$ &c. or (if we put the capital letters A, B, C, D, E, F, G, H, &c. for the ſeveral numeral coefficients, $1, \frac{1}{3}, \frac{1}{9}, \frac{5}{81}, \frac{10}{243}, \frac{22}{729}, \frac{154}{6561}, \frac{2618}{137,781}$, &c. of the terms of the ſeries, reſpectively,) $a^{\frac{1}{3}}\times$ the infinite ſeries $1+\frac{1Ab}{3a}-\frac{2Bb^2}{6a^2}+\frac{5Cb^3}{9a^3}-\frac{8Db^4}{12a^4}+\frac{11Eb^5}{15a^5}-\frac{14Fb^6}{18a^6}+\frac{17Gb^7}{21a^7}-$ &c. in which ſeries both the numerators and the denominators of the generating fractions, $\frac{2}{6}, \frac{5}{9}, \frac{8}{12}, \frac{11}{15}, \frac{14}{18}, \frac{17}{21}$, &c. following the ſecond term, increaſe continually by 3, ſo that it will be eaſy for any one to continue the ſeries to as many terms as he ſhall think proper.

22. In like manner the cube-root of the reſidual quantity $a-b$ is found by the ſame binomial theorem to be

be equal to the infinite ſeries $a^{\frac{1}{3}} - \frac{a^{\frac{1}{3}}b}{3a} - \frac{a^{\frac{1}{3}}b^2}{9a^2} - \frac{5a^{\frac{1}{3}}b^3}{81a^3} - \frac{10a^{\frac{1}{3}}b^4}{243a^4} - \frac{22a^{\frac{1}{3}}b^5}{729a^5} - \frac{154a^{\frac{1}{3}}b^6}{6561a^6} - \frac{2618a^{\frac{1}{3}}b^7}{137,781a^7} -$ &c. or to $a^{\frac{1}{3}} \times$ the infinite ſeries $1 - \frac{b}{3a} - \frac{b^2}{9a^2} - \frac{5b^3}{81a^3} - \frac{10b^4}{243a^4} - \frac{22b^5}{729a^5} - \frac{154b^6}{6561a^6} - \frac{2618b^7}{137,781a^7} -$ &c. or $a^{\frac{1}{3}} \times$ the infinite ſeries $1 - \frac{1Ab}{3a} - \frac{2Bb^2}{6a^2} - \frac{5Cb^3}{9a^3} - \frac{8Db^4}{12a^4} - \frac{11Eb^5}{15a^5} - \frac{14Fb^6}{18a^6} - \frac{17Gb^7}{21a^7} -$ &c. in which ſeries the numeral coefficients of the ſeveral terms are the ſame as in the ſeries that expreſſes the cube-root of $a+b$, but the terms which involve the odd powers of b (which in that ſeries are marked with the ſign +, or all added to the firſt term,) are in this latter ſeries marked with the ſign −, and are all to be ſubtracted from the firſt term, as well as the terms which involve the even powers of b, which are to be ſubtracted from the firſt term in both ſerieſes.

PROBLEM.

23. *Let it now be required to reſolve the firſt caſe of the cubick equation* $x^3 - qx = r$, *in which* r *is greater than* $\frac{2q\sqrt{q}}{3\sqrt{3}}$, *or* $\frac{rr}{4}$ *is greater than* $\frac{q^3}{27}$, *by means of an infinite ſeries derived from the expreſſions given by* CARDAN'S *rule.*

SOLUTION.

We have ſeen in Art. 11. that, if ss be put $= \frac{rr}{4} - \frac{q^3}{27}$, the value of x in this equation will be $= \sqrt[3]{\frac{r}{2} + s} +$

$\sqrt{}^3$

$\sqrt[3]{\frac{r}{2}-s}$. For the sake of avoiding fractions, let e be put $=\frac{r}{2}$. And we shall have $x=\sqrt[3]{e+s}+\sqrt[3]{e-s}$. But (by Art. 21.) $\sqrt[3]{e+s}$ is $=e^{\frac{1}{3}}\times$ the infinite series $1+\frac{s}{3e}-\frac{ss}{9ee}+\frac{5s^3}{81e^3}-\frac{10s^4}{243e^4}+\frac{22s^5}{729e^5}-\frac{154s^6}{6561e^6}+\frac{2618s^7}{137,781e^7}-$ &c; and, by Art. 22. $\sqrt[3]{e-s}$ is $=e^{\frac{1}{3}}\times$ the infinite series $1-\frac{s}{3e}-\frac{ss}{9ee}-\frac{5s^3}{81e^3}-\frac{10s^4}{243e^4}-\frac{22s^5}{729e^5}-\frac{154s^6}{6561e^6}-\frac{2618s^7}{137,781e^7}-$ &c. Therefore $\sqrt[3]{e+s}+\sqrt[3]{e-s}$ is equal to $e^{\frac{1}{3}}\times$ the sum of these two serieses, that is, to $e^{\frac{1}{3}}\times$ the infinite series $2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-$ &c; and consequently the root of the equation $x^3-qx=r$ is $=e^{\frac{1}{3}}\times$ the infinite series $2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-$ &c. *ad infinitum*. Q. E. F.

24. *Note*. This series must always converge, because ss, or $\frac{rr}{4}-\frac{q^3}{27}$, is always less than $\frac{rr}{4}$, or ee. And, when ss is considerably less than ee, or $\frac{rr}{4}-\frac{q^3}{27}$ is considerably less than $\frac{rr}{4}$, or $\frac{rr}{4}$ is very little greater than $\frac{q^3}{27}$, the convergency of the terms of this series will be sufficient to make it useful. But in other cases, when $\frac{rr}{4}$ is much greater than $\frac{q^3}{27}$, (as when it is triple, quadruple or quintuple of it, or still greater,) the terms of this series will converge so slowly as to render it very unfit for practice. And indeed in the most favourable cases it will, as I believe, be less convenient in practice than the expression

$\sqrt[3]{e+s}+\sqrt[3]{e-s}$, or $\sqrt[3]{\frac{r}{2}+s}+\sqrt[3]{\frac{r}{2}-s}$, from which it was derived. However, that it may appear that this ſeries will exhibit the root of the equation $x^3-qx=r$ truly, if we will take the neceſſary pains of computing it, I will here ſubjoin one example, and no more, of the reſolution of a cubick equation of that form by means of it, having taken care to chuſe ſuch numbers for q and r as ſhall make $\frac{rr}{4}$ be but little greater than $\frac{q^3}{27}$, and conſequently ſhall give us only a ſmall number for the fraction $\frac{ss}{ee}$, by the continual multiplication of which the terms of the ſeries are generated.

An example of the reſolution of a cubick equation of the aforeſaid form, $x^3-qx=r$, in the firſt caſe of it, in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, by means of the expreſſion $e^{\frac{1}{3}}\times$ the infinite ſeries $2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-$ &c. obtained in Art. 23.

25. Let it be required to reſolve the equation $x^3-300x=2108$ by means of the infinite ſeries $e^{\frac{1}{3}}\times\overline{2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-\&c.}$ obtained in Art. 23. by the help of Sir ISAAC NEWTON's binomial theorem.

Here

Here q is = 300, and r is = 2108. Therefore $\frac{2q\sqrt{q}}{3\sqrt{3}}$ is $= \frac{2 \times 300 \times \sqrt{300}}{3 \times \sqrt{3}} = \frac{2 \times 100 \times \sqrt{300}}{\sqrt{3}} = \frac{2 \times 100 \times 10\sqrt{3}}{\sqrt{3}} = 2 \times 100 \times 10 = 2000$, which is leſs than 2108, or r. Therefore this equation comes under the caſe of CARDAN's rule, and conſequently may be reſolved by means of the infinite ſeries $e^{\frac{1}{3}} \times \sqrt[3]{2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6}}$ — &c. if that ſeries has been juſtly derived from the third expreſſion of the value of x given by CARDAN's rule.

26. Now, ſince r is = 2108, $\frac{r}{2}$, or e, will be = 1054, and $\frac{rr}{4}$, or ee, will be = 1,110,916. And, ſince q is = 300, $\frac{q}{3}$ will be = 100, and $\frac{q^3}{27}$, or the cube of $\frac{q}{3}$, will be = 1000,000; and conſequently ss, or $\frac{rr}{4} - \frac{q^3}{27}$, will be (= 1,110,916 − 1000,000) = 110,916. Therefore, the fraction $\frac{ss}{ee}$ is $= \frac{110,916}{1,110,916} = .0998$. Therefore $\frac{s^4}{e^4}$ is $= \overline{.0998}|^2 = .009,950$, and $\frac{s^6}{e^6}$ is $= \overline{.0998}|^3 = .000,992$; and $\frac{2ss}{9ee}$ is $= \frac{2}{9} \times .0998 = \frac{.1996}{9} = .022,177$; and $\frac{20s^4}{243e^4}$ is $= \frac{20}{243} \times .009,950 = \frac{.199,000}{243} = .000,818$; and $\frac{308s^6}{6561e^6}$ is $= \frac{308}{6561} \times .000,992 = \frac{.305,536}{6561} = .000,046$; and conſequently, $\frac{2ss}{9ee} + \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6}$ is $= .022,177 + .000,818 + .000,046 = .023,041$; and $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6}$ is $= 2 - .023,041$

=

$= 1.976,959$. But e is $= 1054$. Consequently, $e^{\frac{1}{3}}$, or $\sqrt[3]{e}$, is $= \sqrt[3]{1054} = 10.1768$. Therefore $e^{\frac{1}{3}} \times$ the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} -$ &c. is $= 10.1768 \times 1.976,959 = 20.119,116$. Therefore the root of the equation $x^3 - 300x = 2108$ is $= 20.119,116$. Q. E. I.

27. This value of x is true to five places of figures, the more accurate value of it being 20.119,053, as will easily appear by prosecuting it to three or four more places of figures by Mr. RAPHSON's method of approximation.

28. That 20.119 is very nearly equal to, but somewhat less than, the true value of x in the equation $x^3 - 300x = 2108$, will appear by substituting it instead of x in the left-hand side of that equation. For, if we take $x = 20.119$, we shall have $xx = 404.774,161$, and $x^3 = 8143.651,345,159$, and $300x = 6035.700$; and consequently, $x^3 - 300x = 8143.651,345,159, - 6035.700 = 2107.951,345,159$, which is somewhat less than 2108, or the accurate value of $x^3 - 300x$ in the proposed equation $x^3 - 300x = 2108$. Therefore, 20.119 must be nearly equal to, but somewhat less than, the accurate value of x in that equation.

29. It appears therefore from this example, that this expression, $e^{\frac{1}{3}} \times$ the infinite series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6}$ - &c. does truly exhibit the root of the equation

$x^3-qx=r$ in that case of it which falls under CARDAN's rule, or in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.

30. I now proceed to consider the problem which is the principal object of this paper, which is to shew how from the series $e^{\frac{1}{3}} \times \sqrt{2-\frac{2ss}{9ee}-\frac{20s^4}{243s^4}-\frac{308s^6}{6561s^6}-\text{\&c.}}$ we may derive another series, differing from it only in the signs of some of the terms, by which the equation $x^3-qx=r$ may be resolved in that other case of it which does not come under CARDAN's rule, and in which r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$: and this without any mention of either impossible or negative quantities.

P R O B L E M.

To resolve, by means of an infinite series derived from the infinite series $e^{\frac{1}{3}} \times \sqrt{2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-\text{\&c.}}$ *the second case of the cubick equation* $x^3-qx=r$, *in which* r *is less than* $\frac{2q\sqrt{q}}{3\sqrt{3}}$, *or* $\frac{rr}{4}$ *is less than* $\frac{q^3}{27}$.

S O L U T I O N.

31. We have seen that in the first case of the equation $x^3-qx=r$, in which $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, the product

duct of $e\frac{1}{3}$ into the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6}$ – &c. *ad infinitum*, is equal to the root x. Now there are two different ways of computing this series, which (though not equally short and convenient in practice) are nevertheless equally just and true: and therefore they must both produce the same result for the value of the series. The first way of computing it is the common one, which consists of the following processes; to wit, first, to compute the quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, as was done in the foregoing example, art. 26, where $\frac{rr}{4}$ was found to be $= 1,110,916$, and $\frac{q^3}{27}$ to be 1000,000; 2dly, to subtract $\frac{q^3}{27}$ from $\frac{rr}{4}$, in order to get the quantity ss, which is equal to their difference, and which in the foregoing example was 110,916; 3dly, to divide ss by ee, so as to obtain the value of the fraction $\frac{ss}{ee}$; as in the foregoing example we found the fraction $\frac{110,916}{1,110,916}$ to be $= .0998$; 4thly, to compute the powers of the value found for the fraction $\frac{ss}{ee}$; as in the foregoing example we computed those of .0998, and found its square to be .009,950, and its cube to be .000,992; 5thly, to multiply $\frac{ss}{ee}$, and its powers $\frac{s^4}{e^4}$, $\frac{s^6}{e^6}$, &c. into the co-efficients $\frac{2}{9}$, $\frac{20}{243}$, $\frac{308}{6561}$, &c. respectively,

respectively, as in the foregoing example we multiplied .0998 into $\frac{2}{9}$, and .009,950 into $\frac{20}{243}$, and .000,992 into $\frac{308}{6561}$, and found the products to be .022,117, .000,818, and .000,046; and 6thly, to subtract all the products so obtained from 2 the first term of the series. This is the common and the proper way of computing the series $2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-$ &c. when we want to make use of it in practice. But it may also be computed in another manner, which may be described as as follows.

Instead of ss insert the compound quantity $\frac{rr}{4}-\frac{q^3}{27}$ itself, to which ss is equal, in all the terms of it. And it will be thereby converted into the following series, to wit, $2-\frac{2}{9ee}\times\overline{\frac{rr}{4}-\frac{q^3}{27}}-\frac{20}{243e^4}\times\overline{\frac{rr}{4}-\frac{q^3}{27}}^2-\frac{308}{6561e^6}\times\overline{\frac{rr}{4}-\frac{q^3}{27}}^3-$ &c. or (because ee is $=\frac{rr}{4}$, and consequently $e^4=\frac{r^4}{16}$, and $e^6=\frac{r^6}{64}$) $2-\frac{8}{9rr}\times\overline{\frac{rr}{4}-\frac{q^3}{27}}-\frac{320}{243r^4}\times\overline{\frac{rr}{4}-\frac{q^3}{27}}^2-\frac{19712}{6561r^6}\times\overline{\frac{rr}{4}-\frac{q^3}{27}}^3-$ &c. or $2-\frac{8}{9rr}\times\overline{\frac{rr}{4}-\frac{q^3}{27}}-\frac{320}{243r^4}\times\overline{\frac{r^4}{16}-\frac{2rrq^3}{4\times27}+\frac{q^6}{27\times27}}-\frac{19712}{6561r^6}\times\overline{\frac{r^6}{64}-\frac{3r^4q^3}{16\times27}+\frac{3rrq^6}{4\times27\times27}-\frac{q^9}{27\times27\times27}}-$ &c. or (putting the Greek letters α, β, γ, &c. for the numeral co-efficients $\frac{8}{9}$, $\frac{320}{243}$, $\frac{19712}{6561}$, &c. respectively) $2-\frac{\alpha}{rr}\times\overline{\frac{rr}{4}-\frac{q^3}{27}}-\frac{\beta}{r^4}\times\overline{\frac{r^4}{16}-\frac{2rrq^3}{4\times27}+\frac{q^6}{27\times27}}-\frac{\gamma}{r^6}\times\overline{\frac{r^6}{64}-\frac{3r^4q^3}{16\times27}+\frac{3rrq^6}{4\times27\times27}-\frac{q^9}{27\times27\times27}}-$

 &c.

&c. or $2 - \frac{\alpha}{4} + \frac{\alpha q^3}{27rr} - \frac{\beta}{16} + \frac{2\beta q^3}{4 \times 27rr} - \frac{\beta q^6}{27 \times 27 \times r^4} - \frac{\gamma}{64} + \frac{3\gamma q^3}{16 \times 27rr} - \frac{3\gamma q^6}{4 \times 27 \times 27r^4} + \frac{\gamma q^9}{27 \times 27 \times 27r^6}$ - &c. which consists of a much greater number of terms than the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6}$ - &c. from which it is derived, and in which many of the terms are much more complicated than in that former series. Nevertheless, since the compound quantity $\frac{rr}{4} - \frac{q^3}{27}$ is equal to ss, the insertion of it instead of ss in the terms of that former series cannot alter its real value, though it will make it much more difficult to compute. It must therefore be true of the new and complicated series $2 - \frac{\alpha}{4} + \frac{\alpha q^3}{27rr} - \frac{\beta}{16} + \frac{2\beta q^3}{4 \times 27rr} - \frac{\beta q^6}{27 \times 27r^4} - \frac{\gamma}{64} + \frac{3\gamma q^3}{16 \times 27rr} - \frac{3\gamma q^6}{4 \times 27 \times 27r^4} + \frac{\gamma q^9}{27 \times 27 \times 27r^6}$ - &c. as well as of the former series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6}$ - &c. that, if it be multiplied into $e^{\frac{1}{3}}$, or $\sqrt[3]{e}$, or $\sqrt[3]{\frac{r}{2}}$, the series thereby produced will be equal to the value of x in the equation $x^3 - qx = r$, or that, if the said series be cubed, and also multiplied into q, and from its cube the product of its multiplication into q be subtracted, the remainder will be equal to r, or rather will approximate to the value of r, because, as the quantity substituted for x in the compound quantity $x^3 - qx$ is only a part of an infinite series that is equal to x, it is impossible that the said remainder (which is produced by that substitution) should be accurately equal to the whole value of r.

32. By the help of this observation we may from the foregoing series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6}$ – &c. which, being multiplied into $e^{\frac{1}{3}}$, or the cube-root of $\frac{r}{2}$, expresses the value of x in the equation $x^3 - qx = r$, in the first case of that equation, when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, deduce another series resembling the former in the composition of its terms, but differing from it in the signs to be prefixed to some of them, that will likewise (if multiplied into $e^{\frac{1}{3}}$, or the cube-root of $\frac{r}{2}$) express the value of x in the second case of the same equation, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and which cannot be resolved by CARDAN's rules. This may be done as follows.

33. If in this second case of the equation $x^3 - qx = r$ we subtract $\frac{rr}{4}$ from $\frac{q^3}{27}$, and call the remainder ss (as we before put ss for the opposite difference $\frac{rr}{4} - \frac{q^3}{27}$) and then raise the powers of ss, to wit, s^4, s^6, s^8, s^{10}, and also the correspondent powers of its value $\frac{q^3}{27} - \frac{r^4}{4}$, to wit, $\overline{\frac{q^3}{27} - \frac{rr}{4}}\big|^2$, $\overline{\frac{q^3}{27} - \frac{rr}{4}}\big|^3$, $\overline{\frac{q^3}{27} - \frac{rr}{4}}\big|^4$, $\overline{\frac{q^3}{27} - \frac{rr}{4}}\big|^5$, &c. the even powers

powers of the difference $\frac{q^3}{27} - \frac{rr}{4}$, to wit, $\overline{\frac{q^3}{27} - \frac{rr}{4}}|^2$, $\overline{\frac{q^3}{27} - \frac{rr}{4}}|^4$, &c. will consist of the very same terms, or the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, and with the same signs + and − prefixed to them, as were before contained in the even powers of the opposite difference $\frac{rr}{4} - \frac{q^3}{27}$, when $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$. Thus, for example, the square of $\frac{rr}{4} - \frac{q^3}{27}$ in the former case was $\frac{r^4}{16} - \frac{2rrq^3}{4 \times 27} + \frac{q^6}{27 \times 27}$; and in the present case the square of $\frac{q^3}{27} - \frac{rr}{4}$ is $\frac{q^6}{27 \times 27} - \frac{2q^3rr}{27 \times 4} + \frac{r^4}{16}$, which consists of the same terms, and with the same signs prefixed to them, as were contained in the square of $\frac{rr}{4} - \frac{q^3}{27}$, and differs from it only in the order in which the extreme terms $\frac{r^4}{16}$ and $\frac{q^6}{27 \times 27}$ are placed. And the same observation is true concerning all the other even powers of the opposite differences $\frac{rr}{4} - \frac{q^3}{27}$ and $\frac{q^3}{27} - \frac{rr}{4}$.

Also the odd powers of the difference $\frac{q^3}{27} - \frac{rr}{4}$, to wit, $\frac{q^3}{27} - \frac{rr}{4}$ itself, and $\overline{\frac{q^3}{27} - \frac{rr}{4}}|^3$, $\overline{\frac{q^3}{27} - \frac{rr}{4}}|^5$, &c. will consist of the same terms, or of the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, as were contained in the same odd powers of the opposite difference

rr

$\frac{rr}{4}-\frac{q^3}{27}$, when $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$. But the signs prefixed to the said terms will be contrary to those which were prefixed to them in the former case. Thus, the cube of $\frac{rr}{4}-\frac{q^3}{27}$ in the former case was $\frac{r^6}{64}-\frac{3r^4q^3}{16\times 27}+\frac{3rrq^6}{4\times 27\times 27}-\frac{q^9}{27\times 27\times 27}$; and the cube of $\frac{q^3}{27}-\frac{rr}{4}$ in the present case is $\frac{q^9}{27\times 27\times 27}-\frac{3q^6rr}{27\times 27\times 4}+\frac{3q^3r^4}{27\times 16}-\frac{r^6}{64}$, which consists of the same terms as are contained in the cube of $\frac{rr}{4}-\frac{q^3}{27}$: but they are placed in a contrary order to that in which they stood in the former case; and the signs that are prefixed to them are contrary in every term to what they were before. And the same is true of all the other odd powers of these opposite differences of $\frac{rr}{4}$ and $\frac{q^3}{27}$.

34. It follows, therefore, that if ss be put for $\frac{q^3}{27}-\frac{rr}{4}$ in this latter case of the equation $x^3-qx=r$, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, the even powers of ss, to wit, s^4, s^8, s^{12}, s^{16}, &c. will represent, or be equal to, the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$ in the present case as they represented in the former case, when $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$, and ss was made to stand for $\frac{rr}{4}-\frac{q^3}{27}$; and the several terms represented by the said even powers of ss will have the same signs

signs + and − prefixed to them respectively in this second case as in the first case. And it likewise follows that the odd powers of ss, to wit, ss, s^6, s^{10}, s^{14}, &c. will also represent, or be equal to, the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, as in the former case; but the signs + and − prefixed to the several terms represented by the said odd powers of ss will be contrary to what they were before.

35. If therefore in this second case of the equation $x^3 - qx = r$, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, we put $ss = \frac{q^3}{27} - \frac{rr}{4}$, the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} -$ &c. will represent, or be equal to, a system of terms, derived from the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, that will be the very same in point of composition, that is, will be the very same powers, products, and multiples of $\frac{rr}{4}$ and $\frac{q^3}{27}$, as the terms that were represented by it in the former case, in which $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$: but the terms so represented will not *all* have the same signs + and − prefixed to them as they had before; but those terms in the said system, which are represented by the terms of the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} -$ &c. which involve the even powers of ss, to wit, $\frac{20s^4}{243e^4}$, &c. will have the same signs

prefixed

prefixed to them as they had before when ss stood for $\frac{rr}{4}-\frac{q^3}{27}$; and those terms of the said system which are represented by the terms of the said series which involve the odd powers of ss, to wit, $\frac{2ss}{9ee}$ and $\frac{308s^6}{6561e^6}$, &c. will have contrary signs to those they had before. Consequently, if we change the signs of those terms in the series $2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-$ &c. which involve the odd powers of ss, to wit, the terms $\frac{2ss}{9ee}$ and $\frac{308s^6}{6561e^6}$ &c. the new series thereby produced, to wit, $2+\frac{2ss}{9ee}-\frac{20s^4}{243e^4}+\frac{308s^6}{6561e^6}-$ &c. will represent, or be equal to, a system of terms which will not only be the very same in point of composition (or will be the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$), as those which were represented by the series $2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-$ &c. in the former case, but will also be connected with each other in exactly the same manner by the signs + and −: that is, by Art. 31. the said new series will represent, or be equal to, the following system of terms, to wit, $2-\frac{\alpha}{4}+\frac{\alpha q^3}{27rr}-\frac{\beta}{16}+\frac{2\beta q^3}{4\times 27rr}-\frac{\beta q^6}{27\times 27r^4}-\frac{\gamma}{64}+\frac{3\gamma q^3}{16\times 27rr}-\frac{3\gamma q^6}{4\times 27\times 27r^4}+\frac{\gamma q^9}{27\times 27\times 27r^6}-$ &c.

But

But it has been shewn (in Art. 31.) that, if this system of terms be multiplied into $e^{\frac{1}{3}}$, or the cube-root of $\frac{r}{2}$, and the series thence produced be cubed, and also multiplied into q, and from its cube the product of its multiplication into q be subtracted, the remainder thereby obtained will be (nearly) equal to r. Therefore, if the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} -$ &c. (which represents, or is equal to, the said system of terms, when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and ss is made $= \frac{q^3}{27} - \frac{rr}{4}$,) be multiplied by $e^{\frac{1}{3}}$, or the cube-root of $\frac{r}{2}$, and the series thence produced be cubed, and also multiplied into q, and from the cube of the said series the product of its multiplication into q be subtracted, it will follow that the remainder thereby obtained will be (nearly) equal to r; that is, the product of the multiplication of $e^{\frac{1}{3}}$, or the cube-root of $\frac{r}{2}$, into the infinite series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} -$ &c. is equal to the value of x in the equation $x^3 - qx = r$ in the second case of it, when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$. Q. E. I.

36. This series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} -$ &c. does not always converge, but only when ss is less than ee, or $\frac{q^3}{27} - \frac{rr}{4}$ is less than $\frac{rr}{4}$, or $\frac{q^3}{27}$ is less than $\frac{2rr}{4}$, or $\frac{rr}{4}$ is greater than half

half $\frac{q^3}{27}$, or than $\frac{q^3}{54}$, though less than $\frac{q^3}{27}$. And the nearer $\frac{rr}{4}$ approaches to $\frac{q^3}{27}$, the greater will be the swiftness with which this series will converge.

37. I will now add a few examples of the resolution of cubick equations of the aforesaid form, $x^3-qx=r$, in the second case of those equations, in which r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, by means of the infinite series $e^{\frac{1}{3}} \times \overline{2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.}$ found in Art. 35. in order to confirm the truth of the reasonings by which that series was obtained.

EXAMPLE I.

38. *Let it be required to resolve the equation $x^3-50x=120$ by means of the said infinite series.*

Here q is $=50$; r is $=120$; $\frac{r}{2}$ or e, is $=60$; $\frac{rr}{4}$, or ee, is $=3600$; q^3 is $=125,000$; and $\frac{q^3}{27}$ is $=\frac{125,000}{27}=$ 4629.629,629,629, &c. which is greater than 3600, or $\frac{r}{2}$. Therefore this equation cannot be resolved by CARDAN's rule, but may by the expression $e^{\frac{1}{3}} \times$ the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ provided that series converges. Now, since $\frac{q^3}{27}$ is $=4629.629,629,629$, &c. and

and $\frac{rr}{4}$ is $=3600$, we ſhall have $ss = \frac{q^3}{27} - \frac{rr}{4} = 4629.629,$ $629, 629,$ &c. $- 3600 = 1029.629, 629, 629,$ &c. which is conſiderably leſs than 3600, or ee; and conſequently the ſeries will converge.

39. We ſhall therefore have $\frac{ss}{ee} = \frac{1029.629,629,629}{3600}$ &c. $=.286,00$; and $\frac{s^4}{e^4} = .081,796$; and $\frac{s^6}{e^6} = .023,393$; and conſequently $\frac{2ss}{9ee} = \frac{2 \times .286}{9} = \frac{.572}{9} = .063,55$; and $\frac{20s^4}{243e^4} = \frac{20 \times .081,796}{243} = \frac{1.635,92}{243} = .006,73$; and $\frac{308s^6}{6561e^6} = \frac{308 \times .023,393}{6561} = \frac{7.205,044}{6561} = .001,098$. Therefore $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6}$ is $= 2 + .063,55 - .006,73 + .001,09 = 2.064,64 - .006,73$ $= 2.057,91$. And $e^{\frac{1}{3}}$, or $\sqrt[3]{e}$, is $= \sqrt[3]{60} = 3.914,867$. Therefore $e^{\frac{1}{3}} \times$ the ſeries $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} -$ &c. is $=$ $3.914,867 \times 2.057,91 = 8.0564$; that is, the root of the propoſed equation $x^3 - 50x = 120$ is 8.0564; which is true in three places of figures, the error being in the fourth place of figures, or third place of decimal fractions, where the figure ought to be a 5 inſtead of a 6, the more accurate value of x in that equation being 8.055,810,345,702, as may eaſily be found by Mr. RAPHSON's method of approximation. But 8.0564, the value of x found by the foregoing proceſs, is ſufficiently near to its more accurate value 8.055,810, &c.

to

to shew the truth of the foregoing reasonings. Their difference is only $\frac{6}{10,000}$ parts of an unit, which is only the 13426th part of 8.055,810, &c. or the true value of x.

40. N. B. This equation $x^3-50x=120$ expresses the relation between the diameter of a circle and three chords in it that lie contiguous to each other, and together take up a semicircle, and form a trapezium of which the diameter of the circle is the fourth side. For if the three chords are called b, k and t, and the diameter of the circle is called x, the relation between them will be expressed by the cubick equation $x^3 \left.\begin{matrix} -bb \\ -kk \\ -tt \end{matrix}\right\} \times x = 2bkt$, which, if the numbers 3, 4 and 5 are substituted instead of the letters b, k, and t, will become $x^3-50x=120$. See Sir ISAAC NEWTON's Arithmetica Universalis, Edit. 2d. 1722, page 101.

EXAMPLE II.

41. *Let it be required to find by means of the same series the root of the equation* $x^3-x=\frac{1}{3}$.

Now in this equation q is $=1$, r is $=\frac{1}{3}$, $\frac{r}{2}$ is $=\frac{1}{6}$, $\frac{rr}{4}$, or ee, is $=\frac{1}{36}$, and $\frac{q^3}{27}$ is $=\frac{1}{27}$, which is greater than $\frac{1}{36}$

6 A 2 or

or $\frac{rr}{4}$. Therefore this equation cannot be reſolved by CARDAN's rule, but may by the ſeries $e^{\frac{1}{3}} \times \overline{2 + - + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c.}$ in caſe that ſeries is a converging one.

Now, ſince $\frac{q^3}{27}$ is $= \frac{1}{27}$, and $\frac{rr}{4}$ is $= \frac{1}{36}$, we ſhall have ss, or $\frac{q^3}{27} - \frac{rr}{4}$, $= \frac{1}{27} - \frac{1}{36} = \frac{36-27}{27 \times 36} = \frac{9}{27 \times 36} = \frac{1}{3 \times 36}$, which is leſs than $\frac{1}{36}$, or ee, in the proportion of 1 to 3. Conſequently the ſeries $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ and the ſeries $e^{\frac{1}{3}} \times \overline{2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561s^6} - \&c.}$ will converge. Therefore the equation $x^3 - x = \frac{1}{3}$ may be reſolved by the means of it as follows.

42. Since ss is $= \frac{1}{3 \times 36}$, and $\frac{rr}{4}$, or ee, is $= \frac{1}{36}$, we ſhall have $\frac{ss}{ee} = \frac{1}{3} = .333,333$, and $\frac{s^4}{e^4} = \frac{1}{9} = .111,111$, and $\frac{s^6}{e^6} = \frac{1}{27} = .037,037$, and conſequently $\frac{2ss}{9ee} = \frac{2 \times .333333}{9} = \frac{.666666}{9} = .074,074$, and $\frac{20s^4}{243e^4} = \frac{20 \times .111,111}{243} = \frac{2.222,222}{243} = .009,144$, and $\frac{308s^6}{6561e^6} = \frac{308 \times .037,037}{6561} = \frac{11.407,396}{6561} = .001,738$. Therefore $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ is $= 2 + .074,074 - .009,144 + .001,738 = 2.075,812 - .009,144 = 2.066,668$. And $\sqrt[3]{e}$ is $= \sqrt[3]{\frac{1}{6}} = \frac{1}{\sqrt[3]{6}} = \frac{1}{1.817,121}$. Therefore $e^{\frac{1}{3}} \times$ the ſeries $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ is $= \frac{1}{1.817,121} \times 2.066,668 =$

1.13733;

1.13733; that is, the root of the propofed equation $x^3-x=\frac{1}{3}$ is 1.137,33; which is true to four places of figures, the error being in the fifth place of figures, or the fourth place of decimal fractions, where the figure ought to be an unit inftead of a 3, the more accurate value of x being 1.137,158,164, which differs from the value of it here found by lefs than .00017, or $\frac{17}{100,000^{th}}$ parts of an unit, which is lefs than the 6689th part of 1.137,158,164, or the true value of x.

EXAMPLE III.

43. *Let it be required to find the root of the equation* $x^3-5x=4$.

Here q is $=5$; r is $=4$; $\frac{r}{2}$, or e, is $=2$; $\frac{rr}{4}$, or ee, is $=4$; q^3 is $=125$, and $\frac{q^3}{27}$ is $=\frac{125}{27}=4.629,629,629$, &c. which is greater than 4, or $\frac{rr}{4}$. Therefore this equation cannot be refolved by CARDAN's rule, but may by the infinite feries $e^{\frac{1}{3}}\times\overline{2+\frac{2ss}{9ee}-\frac{20s^4}{243e^4}+\frac{308s^6}{6561e^6}-\&c.}$ in cafe that feries is a converging one.

Now, fince $\frac{q^3}{27}$ is 4.629,629,629, &c. and $\frac{rr}{4}$ is $=4$, we fhall have $\frac{q^3}{27}-\frac{rr}{4}$, or ss, $=.629,629,629$, &c. which is is less than 4, or ee, in the proportion of about 6 to 40, which is a pretty large proportion of minority, and much larger than the proportion of ss to ee in either of the former examples. Consequently the series $e^{\frac{1}{3}} \times \overline{2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.}$ will converge with a greater degree of swiftness than in either of those examples. Therefore the equation $x^3 - 5x = r$ may be resolved by it as follows.

44. Here $\frac{ss}{ee}$ is $= \frac{.629,629, \&c.}{4} = .157,407$; and consequently $\frac{s^4}{e^4}$ is $= .024,777$, and $\frac{s^6}{e^6}$ is $= .003,900$. Therefore $\frac{2ss}{9ee}$ is $= \frac{2 \times .157,407}{9} = \frac{.314,814}{9} = .034,979$, and $\frac{20s^4}{243e^4}$ is $= \frac{20 \times .024,777}{243} = \frac{.495,540}{243} = 002,039$, and $\frac{308s^6}{6561e^6}$ is $= \frac{308 \times .003,900}{6561} = \frac{1.201,200}{6561} = .000,182$, and consequently $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6}$ is $= 2 + .034,979 - .002,039 + .000,182 = 2.035,161 - .002,039 = 2.033,122$. And $e^{\frac{1}{3}}$, or $\sqrt[3]{e}$, is $= \sqrt[3]{2} = 1.259,921$. Therefore $e^{\frac{1}{3}} \times$ the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ is $= 1.259,921 \times 2.033,122 = 2.561,573$; that is, the root of the proposed equation $x^3 - 5x = 4$ is $2.561,573$; which is true to five places of figures, the error being in the sixth place of figures, or the fifth place of decimal fractions, where the figure ought to

6 be

be a 5 instead of a 7. For the accurate value of x in this equation is $\frac{1+\sqrt{17}}{2}$, or $\frac{1+4.123,106}{2}$, or $\frac{5.123,106}{2}$, or 2.561,553; which differs from 2.561,573, or the value of x found by the foregoing series, by only $\frac{20}{1000,000\text{th}}$, or $\frac{2}{100,000\text{th}}$, parts of an unit, or less than the 128,000th part of 2.561,553, or the value of x itself; which is a great degree of exactness.

45. *Note.* That x, or the root of the equation $x^3-5x=r$, is accurately equal to $\frac{1+\sqrt{17}}{2}$, will appear by substituting $\frac{1+\sqrt{17}}{2}$ instead of x in the compound quantity x^3-5x, and observing that it will make that quantity become equal to 4. For, if x is $=\frac{1+\sqrt{17}}{2}$, we shall have $x^3=\frac{1+3\times\sqrt{17}+3\times17+17\times\sqrt{17}}{8}=\frac{52+20\times\sqrt{17}}{8}=\frac{13+5\sqrt{17}}{2}$, and $5x=\frac{5+5\sqrt{17}}{2}$, and consequently $x^3-5x=\frac{13+5\sqrt{17}}{2}-\frac{5-5\sqrt{17}}{2}=\frac{8}{2}=4$. Therefore $\frac{1+\sqrt{17}}{2}$ is $=x$. Q. E. D.

46. These examples sufficiently prove that the expression $e^{\frac{1}{3}}\times$ the series $2+\frac{2ss}{9ee}-\frac{20s^4}{243e^4}+\frac{308s^6}{6561e^6}-$ &c. (which we derived from the other series $e^{\frac{1}{3}}\times\overline{2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-\&c.}$ by the peculiar train of reasoning used in Art. 33, 34, and 35,) gives the true root of the cubick equation $x^3-qx=r$

$qx = r$ in the ſecond caſe of it, in which r is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, and which therefore cannot be reſolved by CARDAN's rule.

I will, however, ſubjoin one more example to the ſame purpoſe; which ſhall be that of the equation $x^3 - 63x = 162$, which both Dr. WALLIS and Mr. DE MOIVRE have reſolved by extracting what they call the impoſſible cube-roots of the impoſſible binomial quantities $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$. Now this equation may be reſolved by the foregoing expreſſion $e^{\frac{1}{3}} \times$ the ſeries $2 + \frac{2ss}{9cc} - \frac{20s^4}{243c^4} + \frac{308s^6}{6561c^6} -$ &c. in the manner following.

EXAMPLE 4.

47. *Let it be required to find the root of the equation*

$$x^3 - 63x = 162.$$

Here q is $= 63$; r is $= 162$; $\frac{r}{2}$, or e, is $= 81$; $\frac{rr}{4}$, or ee, is $= 6561$; $\frac{q}{3}$ is $= 21$; and $\frac{q^3}{27}$ is $= 9261$, which is greater than 6561, or $\frac{rr}{4}$. Therefore this equation cannot be reſolved by CARDAN's rule, but may by the infinite ſeries $e^{\frac{1}{3}} \times \overline{2 + \frac{2ss}{9cc} - \frac{24s^4}{243c^4} + \frac{308s^6}{6561c^6} - \&c.}$ in caſe that ſeries is a converging one.

Now,

Now, since $\frac{q^3}{27}$ is $= 9261$, and $\frac{rr}{4}$ is $= 6561$, we shall have $\frac{q^3}{27} - \frac{rr}{4}$, or ss, $= 2700$, which is less than 6561, or ee, in the proportion of 100 to 243. Consequently the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6461e^6} - \&c.$ and the product of that series multiplied into $e^{\frac{1}{3}}$, or the series $e^{\frac{1}{3}} \times \overline{2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.}$ will converge. Therefore the equation $x^3 - 63x = 162$ may be resolved by it as follows.

48. Since ss is $= 2700$, and $\frac{rr}{4}$, or ee, is $= 6561$, we shall have $\frac{ss}{ee} = \frac{2700}{6561} = \frac{100}{243} = .411,522$, and $\frac{s^4}{e^4} = .169,350$, and $\frac{s^6}{e^6} = .069,691$, and consequently $\frac{2ss}{9ee} = \frac{2 \times .411,522}{9} = \frac{.823,044}{9} = .091,449$, and $\frac{20s^4}{243e^4} = \frac{20 \times .169,350}{243} = \frac{3.387,0}{243} = .013,938$, and $\frac{308s^6}{6561e^6} = \frac{308 \times .069,691}{6561} = \frac{21.464,828}{6561} = .003,271$. Therefore $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ is $= 2 + .091,449, - .013,938, + .003,271 - \&c. = 2.094,720, - .013,938 - \&c. = 2.080,782, - \&c.$ And $e^{\frac{1}{3}}$, or $\sqrt[3]{e}$, is $= \sqrt[3]{81} = 4.326,749$. Therefore $e^{\frac{1}{3}} \times$ the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ is $= 4.326,749 \times 2.080,782, - \&c. = 9.003,021 - \&c.$; that is, the root of the proposed equation $x^3 - 63x = 162$ is $= 9.003,021, - \&c.$ or somewhat less than 9.003,021; which is true to three places of

of figures, the error being in the fourth place of figures, or the third place of decimal fractions, where there ought to be a cypher inſtead of a 3, becauſe the accurate value of x in this equation is 9, as will appear upon trial: for, if x be taken $= 9$, we ſhall have $x^3 = 729$, and $63x = 567$, and conſequently $x^3 - 63x$ $(= 729 - 567) = 162$.

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49. This reſolution of the equation $x^3 - 63x = 162$ anſwers to Dr. WALLIS's reſolution of it by extracting the cube-roots of the impoſſible binomial quantities $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$, inaſmuch as both reſolutions are originally derived from CARDAN's rule. But the difference between them is, that the method here delivered is intelligible in every ſtep of it, whereas Dr. WALLIS's method treats of impoſſible quantities, or quantities of which no clear idea can be formed, in the whole courſe of the proceſs, though it concludes with a reſult that is intelligible, by means of the equality of the impoſſible members of the two ultimate quantities $\frac{9}{2} + \frac{1}{2}\sqrt{-3}$ and $\frac{9}{2} - \frac{1}{2}\sqrt{-3}$ (whoſe ſum is equal to the value of x), and the contrariety of the ſigns + and −, which are prefixed to them. The doctor's method of finding $\frac{9}{2} + \frac{1}{2}\sqrt{-3}$ and $\frac{9}{2} - \frac{1}{2}\sqrt{-3}$ to be the cube-roots

of the impoffible binomial quantities $81+\sqrt{-2700}$ and $81-\sqrt{-2700}$ is only tentative. But Mr. DE MOIVRE has given a *certain* method of finding the cube-roots of fuch quantities in all cafes; but not without the trifection of an angle, or finding (by the help of a table of fines, or otherwife) the cofine of the third part of a circular arc whofe cofine is given; by means of which trifection it is well known (independently of CARDAN's rule, or Mr. DE MOIVRE's procefs) that the fecond cafe of the cubick equation $x^3-qx=r$ (in which $\frac{rr}{4}$ is lefs than $\frac{q^3}{27}$, may be refolved. So that Mr. DE MOIVRE's method of doing this bufinefs, though more perfect than Dr. WALLIS's, does not feem to be of much ufe in the refolution of thefe equations. And both methods are equally liable to the objection above-mentioned, of exhibiting to our eyes, during the whole courfe of the proceffes, a parcel of algebraick quantities, of which our underftandings cannot form any idea; though, by means of the ultimate exclufion of thofe quantities, the refults become intelligible and true. It is by the introduction of fuch needlefs difficulties and myfteries into algebra (which, for the moft part, take their rife from the fuppofition of the exiftence of negative quantities,

 or

or quantities leſs than nothing, or of the poſſibility of ſubtracting a greater quantity from a leſſer), that the otherwiſe clear and elegant ſcience of algebra has been clouded and obſcured, and rendered diſguſting to numbers of men of a juſt taſte for reaſoning; who are apt to complain of it, and deſpiſe it, on that account. And, doubtleſs, they have too much reaſon to do ſo, and to ſay, in the words of the famous *Monſieur* DES CARTES in his diſſertation *De Methodo*, page 11, *Algebram verò, ut ſolet doceri, animadverti certis regulis et numerandi formulis ita eſſe contentam, ut videatur potiùs ars quædam confuſa, cujus uſu ingenium quodammodò turbatur et obſcuratur, quam ſcientia, quâ excolatur et perſpicacius reddatur.* If this complaint was juſt in DES CARTES's time, there is certainly much more reaſon for it now.

50. The paſſage above alluded to in Dr. WALLIS's algebra, is in the 48th chapter, pages 179, 180, of the folio edition at London in 1685. And Mr. DE MOIVRE's method of extracting the cube-root of an impoſſible binomial quantity, as $81+\sqrt{-2700}$, or $a+\sqrt{-b}$, is publiſhed in the appendix to the ſecond volume of profeſſor SAUNDERSON's algebra, pages 744, 745, 746, 747. It is very ingenious, and ſhews that author's great ſkill in the uſe and management of al-

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gebraick quantities. See alſo on this ſubject CLAIRAUT's Elémens d'Algébre, Part V. Section 9. pages 286, 287, 288, and a paper of Monſieur NICOLE in the memoirs of the French Academy of Sciences for the year 1738, pages 99 and 100. See alſo MACLAURIN's algebra, Part I. the ſupplement to the 14th Chapter, pages 127, 128, 129, 130; and the Philoſophical Tranſactions, N°. 451.

51. If any gentleman ſhould be inclined to compute the ſeries $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6}$ − &c. to more than four terms, he will find the firſt eight terms of it to be as follows, to wit, $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \frac{1870s^8}{59049e^8} + \frac{111,826s^{10}}{4,782,969e^{10}} - \frac{2,358,512s^{12}}{129,140,163e^{12}} + \frac{120,646,960s^{14}}{8,135,830,269e^{14}}$.